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THE "NO-ROLLING" CURVES OF AMSLER'S PLANIMETER.

BY F. MORLEY.

1. The Planimeter. I shall first restate the usual explanation of Amsler's instrument in kinematical language. According to my experience, kinematical intuitions should be freely used in elementary courses on Calculus.

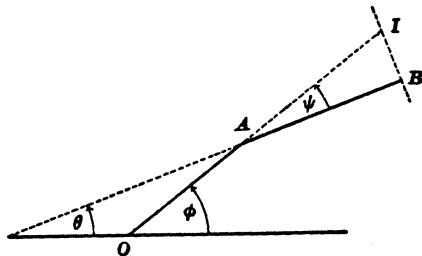


FIG. 1.

OA, AB (fig. 1) are rods of lengths a and b . They are hinged at A , and O is fixed. We suppose that B describes a circuit and that the rods oscillate back to their old position. And we assume that the area χ is known to be

$$\frac{1}{2} \int pv \, dt,$$

where pv is the moment of the velocity about O .

The velocity of B is made up of

$$\begin{aligned} a\dot{\theta} & \text{ perpendicular to } OA, \\ b\dot{\phi} & \text{ " " } OB. \end{aligned}$$

The moments about O of these are

$$a\dot{\theta}(a + b \cos \psi) \text{ and } b\dot{\phi}(a \cos \psi + b).$$

The sum of these is the resultant moment. Hence

$$pv = a^2\dot{\theta} + b^2\dot{\phi} + ab(\dot{\theta} + \dot{\phi})\cos \psi.$$

Hence the area is

$$\chi = \frac{1}{2} \int [a^2\dot{\theta} + b^2\dot{\phi} + ab(\dot{\theta} + \dot{\phi})\cos \psi] \, dt;$$

or, since the change of θ and of ϕ is 0,

$$\chi = \frac{1}{2} ab \int (\dot{\theta} + \dot{\phi}) \cos \psi \, dt;$$

or, since $\dot{\psi} = \dot{\theta} - \dot{\phi}$,

$$\begin{aligned} \chi &= \frac{1}{2} ab \int (2\dot{\theta} - \dot{\psi}) \cos \psi \, dt \\ &= ab \int \dot{\theta} \cos \psi \, dt. \end{aligned}$$

Now a wheel placed on AB at a distance c from A has, perpendicular to AB , a velocity

$$a\dot{\theta} \cos \psi + c\dot{\phi},$$

and therefore the record of the wheel is

$$\begin{aligned} &\int (a\dot{\theta} \cos \psi + c\dot{\phi}) \, dt, \\ &\text{or } a \int \dot{\theta} \cos \psi \, dt. \end{aligned}$$

Hence the area described is

$$b \times \text{record of the wheel.}$$

2. The curves. In a second mode of explanation, indicated in Williamson's *Integral Calculus* and suggested by Sir Robert Ball, use is made of the curves for which the wheel slides without rolling. On examining these curves I find some facts which appear worth mentioning.

I suppose the wheel now to be at the point B , and the curves are those described when the velocity of B across AB is zero, or

$$a\dot{\theta} \cos \psi + b\dot{\phi} = 0.$$

These curves are the "orthogonal trajectories" of the curves for which B moves at right angles to AB , that is of the ring of circles of radius b whose centres lie on a circle of radius a . Thus, as a geometric definition, our curves are *the orthogonal curves of a ring of circles*. Or, in analogy with the orthogonal trajectory of a row of equal circles, we can call such a curve *the tractrix of a circle*.

If we invert the ring of circles as to its centre O , we have again a ring of

circles, and the old orthogonal curves pass into the new orthogonal curves. Thus the inverse of a tractrix as to O is again a tractrix.

In the case when $b = a$, the ring of circles inverts into the tangents of a circle, and the orthogonal curve of the tangents is the involute of the circle. Thus this special tractrix is *the inverse of the involute of a circle, as to the centre of the circle*.

3. Stereographic projection of the curves. To generalize the last remark, consider the two concentric circles which touch all the circles of the ring. Their radii are $r = a + b$, $r_1 = |a - b|$. Invert the plane into a sphere of diameter $\sqrt{rr_1}$, touching the plane at the origin O . The two concentric circles thus become equal and parallel circles of the sphere, of radius $\frac{\sqrt{rr_1}}{r + r_1}$. Call these *latitude circles*. The ring becomes a system of circles on the sphere, touching the latitude circles. When $b < a$, this system is a ring of small circles, whose planes all touch a right cylinder through the latitude circles, and no ground is gained. But when $b > a$, the system is a ring of great circles, whose planes all touch a right cone through the latitude circles.

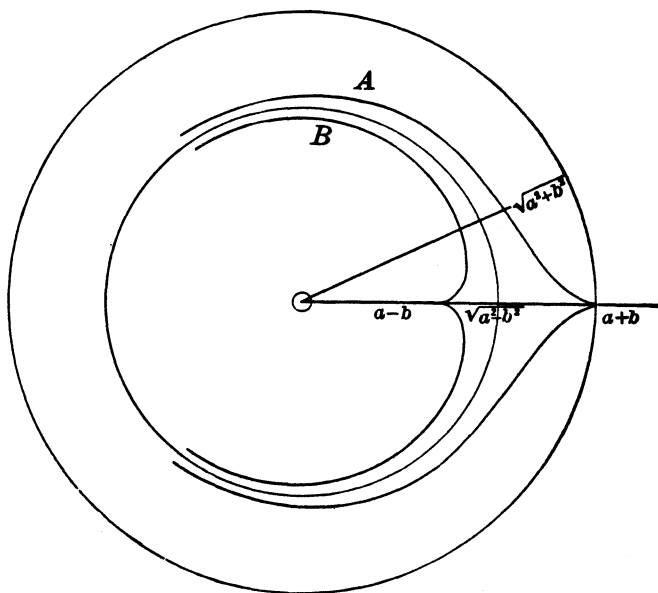


FIG. 2.

Take then the case $b > a$. The tractrix in the plane maps into the orthogonal curve of the tangent planes of a right cone, that is *into the spherical involute*

of a *latitude circle*, or curve traced by a point of a circle, whose centre is fixed, which rolls on a plane.

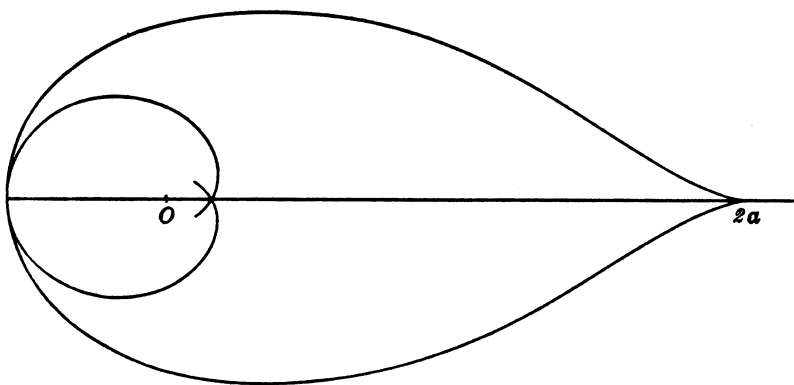


FIG. 3.

Since each point of the rolling circle is moving at right angles to its plane, the curve on the sphere is a *curve of constant slope*, that is the tangent makes always a constant angle with a given line.

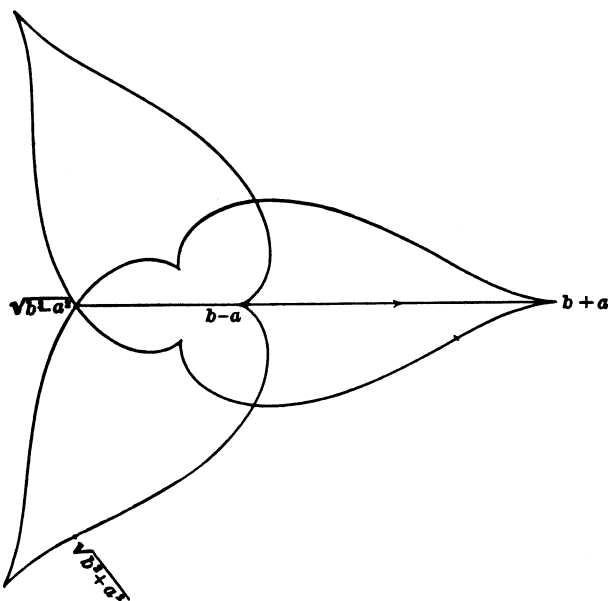


FIG. 4.

4. The form of the tractrix of a circle. The three cases, $b < a$, $b = a$, $b > a$ are illustrated in the figures. In (2) there is an asymptotic circle of

radius $\sqrt{a^2 - b^2}$. In (3) the asymptotic circle reduces to a point. In (4), the case of most interest, we have equidistant cusps arranged along the circles $b \pm a$. The angular distance from cusp to cusp is determined at once from the stereographic projection (§3). For the great circle, whose perimeter is $\pi \sqrt{b^2 - a^2}$, rolls on the latitude circles, whose radius is $(b^2 - a^2)/2b$. Hence from one cusp to the next the angle is

$$\pi b / \sqrt{b^2 - a^2},$$

and the curve is closed when $b/\sqrt{b^2 - a^2}$ is a rational number.

5. The evolute. The instantaneous centre of the motion is I the intersection with OA of the perpendicular to AB through B . The centrode in the moving plane is the line BI . The motion is the rolling of this line on the fixed centrode, which is therefore the evolute of our curve. For the evolute, or locus of I , we have (fig. 1):

$$\text{radius vector } r = OI = c + b \sec \psi,$$

$$\text{perpendicular on tangent } p = b + a \cos \psi,$$

and therefore,

$$(r - a)(p - b) = ab,$$

or

$$a/r + b/p = 1.$$

Also

$$\text{arc } s = BI = b \tan \psi.$$

If now this evolute rolls on the line IB which is fixed, the curve itself will always pass through B , the point A will be fixed, and O will describe a circle whose centre is A . That is, the problem of finding a curve which, when rolled on a line, has as roulette a circle is solved by taking for the curve the evolute of the tractrix of a circle, and for the tracing point the centre of that circle. That this is in fact the only solution is seen at once from the formula:

$$\text{curvature of roulette} = D_p(p/r).$$

6. The Newtonian Catenary. As the evolute of the ordinary or right tractrix is the catenary, it is natural to ask under what force, tending to O or from O , will a string assume the form of the evolute of the tractrix of a circle. From the equations:

$$\text{tension } T \propto 1/p,$$

$$\text{force } F \propto D_r T,$$

we have in our case

$$F \propto D_r (1 - a/r) \propto 1/r^2.$$

Thus the requisite law is that of the inverse square.

Naturally, for attraction that part of the evolute must be taken which is convex to O , for repulsion the concave part; and there can be no inflexions at a finite distance.

The determination of the particular catenary, when the length of the string and the terminals are given, requires a closer study of the curve. We have (§5):

$$r = a + b \sec \psi, \quad (1)$$

$$p = b + a \cos \psi, \quad (2)$$

$$s = b \tan \psi, \quad (3)$$

and $rD_s \theta = \cos \psi$, in any curve, so that here

$$d\theta = \frac{b d\psi}{b + a \cos \psi},$$

$$\frac{\theta}{2b} = \frac{1}{\sqrt{b^2 - a^2}} \tan^{-1} \left(\sqrt{\frac{b-a}{b+a}} \tan \frac{\psi}{2} \right),$$

or if

$$\lambda = \sqrt{b^2 - a^2} / b,$$

$$\tan(\lambda\theta/2) = \sqrt{\frac{b-a}{b+a}} \tan \frac{\psi}{2},$$

$$\cos \lambda\theta = \frac{a + b \cos \psi}{b + a \cos \psi},$$

and finally we have the polar equation

$$\frac{a^2 - b^2}{r} = a - b \cos \lambda\theta. \quad (4)$$

Also

$$\begin{aligned} \sin \lambda\theta &= \frac{\sqrt{b^2 - a^2} \sin \psi}{b + a \cos \psi} \\ &= \lambda s / r, \end{aligned}$$

or

$$\lambda s = r \sin \lambda\theta. \quad (5)$$

The initial position has been that in which the point B is at its maximum distance from O ; for the case when, in the initial position, the rods are doubled up, we change the sign of a . Whether the curves so obtained are to be regarded as distinct or as parts of one and the same curve is to some extent a matter of taste; but I take the latter view, which avoids distinguishing the cases when λ is rational, irrational, and imaginary.

The equations (4) and (5) serve to determine the position of the apse line and the values of a and b when the terminals, the length of string, and (in the case of repulsion) the number of times the curve is to enclose the centre, are given.

For given terminals and length of string we have in general *two* solutions, one for repulsion and one for attraction. The relation of these is one of considerable intricacy. But in the special case when the terminals are equidistant from the centre, the apse line is the same in both curves. Hence from (5) where s, r, θ are given, and s is less than r , *the two values of λ are the same*. Hence we may put

$$a_1 = \mu a, \quad b_1 = \mu b$$

But from (1) and (3)

$$r^2 + 2ar = b^2 - a^2 + s^2,$$

and

$$r^2 - 2a_1r = b_1^2 - a_1^2 + s^2,$$

or

$$r^2 - 2\mu ar = \mu^2 (b^2 - a^2) + s^2.$$

Therefore

$$(\mu + 1)(r^2 - s^2) = (\mu + \mu^2)(b^2 - a^2),$$

or

$$r^2 - s^2 = (b - a)\mu(b + a)$$

= product of apsidal distances.

That is, *when a string of length $2s$ is attached to two points at a distance r from the centre, and $s < r$, if c be the apsidal distance for attraction, and c_1 that for repulsion, then*

$$cc_1 = r^2 - s^2.$$

7. The form of the evolute. The general form of the curve should be noted; especially as the text-books on strings avoid giving figures.

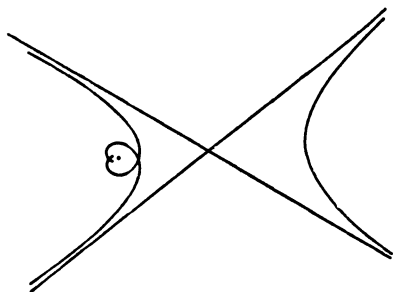


FIG. 5A.

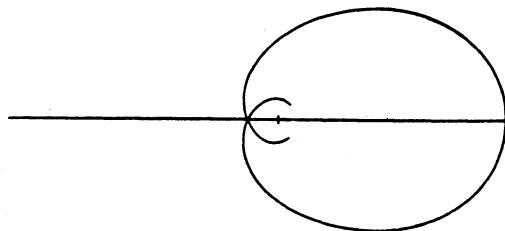


FIG. 5B.

Figs. 5 A and 5 B represent the evolutes of a curve of the form 2; and give a fair idea of the whole class. Fig. 7 is the curve (a circular quartic)

for which $\lambda = \frac{1}{2}$; the general case can be grasped from this by adding more loops round the centre, and by rotating the whole about the centre through

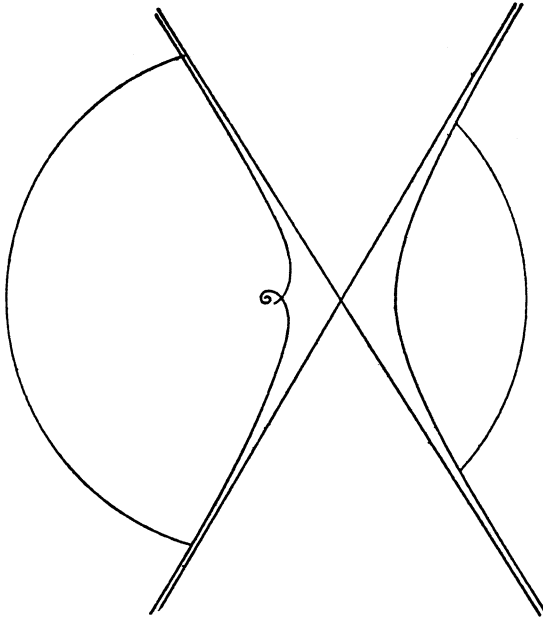


FIG. 6.

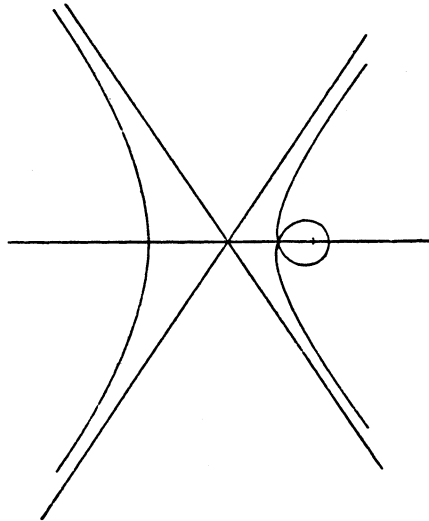


FIG. 7.

equal angles. Fig. 6 is an interesting special case, for which $a = b$, $\lambda = 0$. The equations of §6 take the form

$$2a/r = 1 - \theta^2,$$

$$\theta = \tan \frac{\psi}{2},$$

$$s = r\theta.$$

Thus the length of a symmetric arc of the curve is equal to that of the intercepted arc of a concentric circle, as shown in the figure.

The parallel with the ordinary catenary may be developed; for instance if we ask what is the least length which can rest over two pins equidistant from O , with free ends, under an attraction, it will appear that the curve must be of the form (6) and the least length is

$$2r(1 + \theta)^2,$$

where r is the distance of a pin from the centre, and 2θ is the angle (supposed < 2) which the pins subtend at the centre. This we may leave as an exercise.

The equation $2a/r = 1 - \theta^2$ is obtainable from the theorem at the end of § 6; for if the curve of repulsion be a circle of radius r , then $c_1 = r = s/\theta$, whence the theorem gives

$$cr = r^2 - r^2\theta^2$$

or

$$r(1 - \theta^2) = c.$$

8. Kinematics of the bicycle. The figures of this paper have been in part drawn by students of the college; and one of them, Mr. Redfield, made the interesting remark that the tractrix gives a good idea of the track of the rear wheel of a bicycle when the track of the front wheel is a circle (fig. 2). For small deviations of the rear wheel from the vertical the variation of the wheel-base is in fact very small, however great the inclination of the wheels, in the usual type of machine in which the steering post points (in the normal position) near the point of contact of the front wheel with the ground.

To examine the relative curve of the bicycle tracks, when the rear wheel is held vertical, call p and p_1 the distances from the axles to the steering head, q the distance between p and p_1 . Let θ be the inclination of p to the horizontal, ϕ the angle turned through by p_1 , a the radius of either wheel (the wheels are supposed equal, as in the American type of machine). Then the coordinates of the point of contact of the front wheel, referred to axes through that of the rear wheel, such that z is vertical and x the line of symmetry, are

$$x = p \cos \theta + q \sin \theta + p_1 \cos \phi \cos \theta + \frac{a \sin \theta \cos \theta \sin^2 \phi}{\sqrt{1 - \sin^2 \theta \sin^2 \phi}}$$

$$y = p_1 \sin \phi - \frac{a \sin \theta \sin \phi \cos \phi}{\sqrt{1 - \sin^2 \theta \sin^2 \phi}},$$

$$z = a + p \sin \theta - q \cos \theta + p_1 \sin \theta \cos \phi - a \sqrt{1 - \sin^2 \theta \sin^2 \phi}.$$

Hence, since $z = 0$, we have

$$p_1 \sin \theta \cos \phi + A = a \sqrt{\cos^2 \theta + \sin^2 \theta \cos^2 \phi},$$

where

$$A = a + p \sin \theta - q \cos \theta;$$

or $(a^2 - p_1^2) \sin \theta \cos \phi = Ap_1 + a \sqrt{A^2 - (a^2 - p_1^2) \cos^2 \theta},$

whence, if $\tan \frac{\theta}{2} = t$, $\cos \phi$ is a rational function of t and $\sqrt{Qt}$, where Qt is a quartic in t .

Hence we can take, for the complete discussion of the relative curve, x , $y/\sin \phi$, and therefore also y^2 , as elliptic functions of a parameter u .

HAVERFORD COLLEGE, MAY, 1899.